

COMMON CORE UNIT:

Straw Man Outline for an Intense Engagement Exemplar in Grade 7

UNIT SUMMARY

Content area: **Addition and subtraction of rational numbers**

Associated CCSS content standards: **7.NS.1, 7.NS.3**

Total instructional time: **15 days**

The rational numbers are an arithmetic system that includes 0 as well as positive and negative whole numbers and fractions. Wherever the term “rational numbers” is used, numbers of all types are implied, including fractions in decimal notation.

The straw man outline presented here does not develop the arithmetic of integers before the arithmetic of rational numbers in general. Rather, each specific phase of the outline would likely begin with integers for simplicity and incorporate other rational numbers second. As with any feature of the straw man, bidders may critique this choice and present alternatives if desired.

Prior knowledge assumed. In grade 6, students learned about signed numbers and what kinds of quantities they can be used to represent. They located them on a number line. As a result of this study, students should have come away thinking of the negative side of the number line as being the mirror reflection of the positive side. For example, by reasoning that the reflection of a reflection is the thing itself, they will have learned that $-(-a) = a$. (Here a may be positive, negative, or zero.) Grade 6 students also learned about absolute value and ordering of rational numbers, including in real-world contexts.

In Grade 6, students will have supported their reasoning about rational numbers with models of rational numbers. These models should continue to function in Grade 7 to support the development of rational arithmetic.

As an aside, the grade 6-to-grade 7 progression for rational numbers is structurally similar to the grade 3-to-grade 4 progression for fractions. In grade 3, students first learned about fractions and what kinds of quantities they can be used to represent. They located them on a number line and ordered them. Then, in grade 4, they began doing arithmetic with these new numbers. Likewise, in grade 7, now that we have some new numbers, we again have to learn how to add, subtract, multiply, and divide with them.

Use of models. Colored chips are commonly used; a similar approach uses small plastic plus signs and minus signs in place of colored chips. Such models have the disadvantage that they do not seem to lend themselves well to fraction reasoning, or consequently to multiplication and division of rational numbers. (Perhaps one could represent a rational number such as $-3/5$ by three chips where one chip represents $-1/5$. In a problem with mixed denominators this would become unwieldy.) If chips are used, it may be especially important for contextual illustrations to involve fractional quantities.

A mathematically robust model for a rational number is an arrow on a number line. To represent a number q , an arrow is drawn from 0 to the point q . The length of the arrow represents the absolute value of the number, and the direction of the arrow indicates the sign of the number. Addition combines arrows tip-to-tail, while multiplication stretches or shrinks arrows by the multiplicative factor, and possibly reverses their direction. This fully represents the scaling/resizing nature of multiplication; see 5.NF.5. (Multiplication with chips seems to revert to an equal-groups picture of multiplication more appropriate to Grade 3; see 3.OA.1.) Other models may be possible.

Whatever models are used, students should represent sums and differences of rational numbers on the number line at various points of the engagement. They should come to recognize, or it can be pointed out to them, that $p + q$ is the number located a distance $|q|$ from p , in the positive or negative direction depending on whether q is positive or negative. They should observe, or it should be pointed out to them, that this agrees with addition on the number line as it may have been practiced in earlier grades.

Deductive reasoning, concrete models, and contextual situations. In building up the arithmetic of rational numbers in Grade 7, there is no avoiding a certain amount of deductive reasoning. Students need not deduce everything from definitions and stated assumptions, but neither on the other hand can students reach all of the necessary conclusions by making analogies to everyday situations. That said, everyday situations should certainly be woven into the treatment in order to provide plausibility to results, keep the material interesting, and ensure that students can apply what they learn. In the outline below, a number of calculations are laid out in deductive steps. These calculations are meant to show how the key mathematical ideas fit together. These calculations are not meant to suggest that teachers should fill up the blackboard with equations in the style of a college mathematics course. In each case, one expects that the reasoning will be appropriately motivated and supported by models and contextual situations.

Sequence of ideas and activities

1. Day 1. We review the key grade 6 takeaways, assigning homework with review problems.
2. We work hard on addition for 5 days.
 - a. Day 2. We begin with some simple and plausible principles on which others will be built.

Students are told that addition with signed numbers obeys the same general rules as addition of whole numbers and fractions, including commutativity and associativity. And indeed, **addition of positive numbers agrees with the addition they have been doing all along.**

Examples:

$$(+3) + (+5) = +8 \text{ agrees with } 3 + 5 = 8.$$

$$(+1/2) + (+1/3) = +5/6 \text{ agrees with } 1/2 + 1/3 = 5/6.$$

Notes. This could be approached in any number of ways - as a logical consequence of a formal definition of addition of rational numbers (along the lines of 7.NS.1.b), as an implication of a model for addition of signed numbers (such as chips or arrows), and/or

simply as a useful thing to say in light of various real-world situations that we model with positive and negative numbers. (For example, if a balloon is 100 meters above sea level and it rises 30 meters, then it is 130 meters above sea level.)

Students are then told that another similarity to what they have done before is that **adding 0 to any number leaves the number unchanged**:

Examples:

$$(+1/3) + 0 = +1/3$$

$$0 + (-8.1) = -8.1.$$

Notes. Again this can be approached in any number of ways - as a logical consequence of a formal definition of addition of rational numbers along the lines of 7.NS.1.b, as simple common sense left mostly unremarked upon, as a principle we would like to hang onto going forward into algebra, and/or simply as a useful thing to say in light of various real-world situations that we model with positive and negative numbers. (For example, if someone writes you a check for \$0, you have the same amount of credit or debt you had before.)

Classwork and homework: Students solve a variety of word problems in a context involving positive and negative numbers, where the problem calls for addition of two positive numbers or addition of zero to a number.

b. Day 3. Students then work on problems like:

$$(+10.3) + (-10.3) = 0$$

and

$$(-2/3) + -(-2/3)$$

$$= (-2/3) + (+2/3)$$

$$= 0$$

to learn that **a number and its opposite sum to zero**. Again this can be approached in any number of ways - as a logical consequence of a formal definition of addition of rational numbers along the lines of 7.NS.1.b, as an implication of a model for addition of signed numbers such as chips or arrows, and/or simply as a useful thing to say in light of various real-world situations that we model with positive and negative numbers. (For example, if you win \$10.30 and then lose \$10.30, you haven't gained or lost anything. Another example, which would require appropriate scientific knowledge, is that a positive proton and a negative electron combine to form a neutral atom.)

Classwork and homework: Students work on some pure computational problems, and some word problems, in which a number and its opposite are added. These might include problems that call for the addition of three numbers, two of which are opposites of one another.

- c. Day 4. Students find the **sum of negative numbers** using a combination of common sense, models and/or definitions, and what has been learned.

To find: $(-7) + (-4) = ?$

Students could use a model of addition to find the sum -11 , and also think of the problem contextually: If it is already 7 degrees below zero, and the temperature drops another 4 degrees, how cold is it now? 11 degrees below zero.

Classwork and homework: After doing a few of these, students will catch on to the general pattern for adding two negative numbers. They practice computation problems and word problems that include fractions and decimals, and perhaps problems with three or more negative addends. Practice should also mix in some problems of the types previously encountered.

- d. Day 5. Students find the **sum of a positive number and a negative number**. This is the first really difficult moment in the sequence. Again this is a combination of common sense, models and/or definitions, and what has been learned.

Examples

$$(+700) + (-100)$$

$$= (+600) + (+100) + (-100) \quad \text{because } (+600) + (+100) = +700$$

$$= (+600) + 0 \quad \text{by property of adding opposites}$$

$$= +600 \quad \text{by property of zero.}$$

$$(+50) + (-80) = (+50) + (-50) + (-30) = 0 + (-30) = -30.$$

(This “neutral pair” approach is meant to show where the answer comes from; it is not necessarily an efficient algorithm for adding rational numbers.)

Students are told the **general pattern for the results of addition**, if they haven’t figured it out. When adding two numbers with different signs, the number with smaller absolute value partially cancels the number with larger absolute value. The sum has the same sign as the addend with larger absolute value, but the sum is closer to 0 than that addend is.

- e. Day 6: **Practice**. No new ideas are introduced. Students do computational and word problems that call for adding three or more signed numbers of various signs and relative absolute values, including fractions and decimals. (Students may use number line representations to support reasoning if they like, but the problems themselves do not refer to representations or models.) The two goals are fluency in computation and confidence and correctness in application.

Toward the end of this work, we stop using the + symbol to indicate positive numbers.

3. We now turn to subtraction. Another 4 days of hard thinking.

- a. Day 7. As was the case for addition, we begin with simple and plausible principles on which others are built.

Students are reminded that in their previous study of subtraction, they check their answer to subtraction by adding. In general, we check our answer to $a - b = c$ by adding $b + c$ and making sure we get back to a . This is still true for rational numbers. If ever in doubt about a subtraction, we check our results by adding.

Students are told some important special cases of this: first, **subtracting anything from itself gives zero**: $a - a = 0$. This would probably just strike most students as common sense. It is true because if you add $a + 0$ you get back to a .

Also, **subtracting zero leaves a number unchanged**: $x - 0 = x$. Again plausible, and true because if you add $x + 0$ you get back to x . The teacher does a few numerical examples at the board to make sure people are good with this. The presentation includes contextual examples in addition to the above mathematical reasoning.

These special cases are so plausible that some students might like them perfectly well without completely following the subtle argument about adding back; such students are not browbeaten with the reasoning but are simply allowed to embrace the results willingly.

- b. Day 8. Students learn the general rule for subtraction, $a - b = a + (-b)$. **Subtracting means adding the opposite**.

During their work with addition, students likely already noticed special cases of the general rule. For example, in $(+10) + (-3)$, the answer $+7$ was like subtracting 3 from 10. Students collect these suggestive examples and observe that subtractions familiar from previous grades can be recast as adding the opposite. For example, $4/5 - 1/5 = 3/5$ is familiar from previous grades, and more recently one may recall the result of $4/5 + (-1/5)$ as being $3/5$ as well. So in this case, $4/5 - 1/5 = 4/5 + (-1/5)$. From such examples, supported as appropriate by models and contextual situations, students conjecture, or are led to conjecture, that all subtraction of rational numbers is adding the opposite.

Students make a small test of this conjecture by showing how this rule agrees what came immediately before: namely, it reproduces the prior conclusions that $a - a = 0$ (because $a + (-a) = 0$) and that $x - 0 = x$ (because $x + (-0) = x + 0 = x$).

Examples like the following are then used to motivate/derive/justify the general rule. Each example given might be motivated by a context, then modeled, then formalized; or treated like a logical argument and then modeled and illustrated in context, to “see it in action.”

Example:

$$3 - 5$$

$$= 3 + 0 - 5 \quad \text{by the property of 0}$$

$$= 3 + (-5) + 5 - 5 \quad \text{by the property of adding opposites}$$

$$= 3 + (-5) + "a - a" \quad \text{noticing an instance of one of the special cases above}$$

$$= 3 + (-5) + 0 \quad \text{by the above special case}$$

$$= 3 + (-5) \quad \text{by the property of 0; this step shows the general rule}$$

$$= -2. \quad \text{by previous work.}$$

As with any subtraction problem, we check the answer by addition:

$$5 + (-2)$$

$$= 3. \text{ It checks.} \quad \text{by previous work on adding rational numbers.}$$

Another Example

$$-8 - (-12)$$

$$= -8 + 0 - (-12) \quad \text{by the property of 0}$$

$$= -8 + 12 + (-12) - (-12) \quad \text{by the property of adding opposites}$$

$$= -8 + 12 + "a - a" \quad \text{noticing an instance of one of the special cases}$$

$$= -8 + 12 + 0 \quad \text{by the special case}$$

$$= -8 + 12 \quad \text{by the property of 0; this step shows the general rule}$$

$$= 4. \quad \text{by previous work.}$$

As with any subtraction problem, we check the answer by addition:

$$-12 + 4$$

$$= -8. \text{ It checks.} \quad \text{by previous work on adding rational numbers.}$$

Aside: Here is the first example another way:

$$\begin{aligned}
 & 3 - 5 \\
 &= 3 + 0 - 5 && \text{by the property of 0} \\
 &= 3 + 2 + (-2) - 5 && \text{by the property of adding opposites} \\
 &= 5 + (-2) - 5 && \text{by previous work} \\
 &= 5 - 5 + (-2) && \text{by properties of addition} \\
 &= "a - a" + (-2) && \text{noticing an instance of one of the special cases above} \\
 &= 0 + (-2) && \text{by the above special case} \\
 &= -2. && \text{by previous work.}
 \end{aligned}$$

Compared to the first calculation shown for $3 - 5$, this is probably an easier way to get to the answer, because it takes students there by a reassuringly familiar path of “taking small from big.” However, the point of having this discussion was not to train students in an optimal method for getting the answer to problems like $3 - 5$. The point was to motivate, illustrate, or justify the fact that subtraction is adding the opposite. Observe that in the second method for $3 - 5$, this general rule does not appear.

Classwork and homework: Students practice subtraction of two rational numbers, including fractions and decimals. In some cases they are asked to check the results by addition.

- c. Day 9: Students work with **parentheses**. The first problem they do is $-(8+5) = ?$. This is easy if we begin inside the parentheses, $8+5 = 13$ so $-(8+5) = -13$. Students are shown this result in a new light: because $-13 = -8 + -5$, our result can be written as $-(8+5) = -8 + -5$. This can be described by saying that **“the opposite of a sum is the sum of the opposites.”** This can be illustrated with models and contextual situations. Students verify this for all of the possibilities with signs. The principle is then extended to the case of three or more summands. Students might be invited to think of this as “distributing in the minus sign.” (The use of the term “distributing” will be justified later during the study of multiplication of rational numbers, when we find that -1 times a number is the opposite of the number: $(-1)(a) = -a$. Then the principle that the opposite of a sum is the sum of the opposites can be seen from the distributive property as $-(b + c) = (-1)(b + c) = (-1)b + (-1)c = -b + -c$.)

Next problem: $6 - (8+5) = ?$ We can do this as $6 - (8+5) = 6 - 13 = 6 + (-13) = -7$. In that approach, we evaluate parentheses first as usual and then apply the principle that subtraction is adding the opposite. Or, we can view the term in parentheses as a “chunk” and apply the subtraction principle first: $6 - (8+5) = 6 + -(8+5)$. As a second step, we can evaluate the term in parentheses; this leads to $6 + (-13)$ as before. Alternatively, as a second step we can handle the

term in parentheses by applying the principle that the opposite of the sum is the sum of the opposites:

$$6 + -(8+5) = 6 + (-8) + (-5) .$$

Now we have a sum of three summands, $6 + (-8) + (-5)$. This can be evaluated in various ways, e.g., adding the first two terms first gives $(-2) + (-5) = -7$.

Classwork and homework: Students solidify this by evaluating simple numerical expressions with sums and differences of positive and negative rational numbers that involve parentheses.

- d. Day 10: **Practice.** Students do computational problems involving addition and subtraction of three or more rational numbers, including positive and negative fractions and decimals, problems with parentheses, and problems with more than one subtraction operation. The goal is fluency.

4. We now turn to finding distances on the number line.

Students in Grade 6 learned that the absolute value of a number is its distance from 0 on a number line. They also found the distance between two points in the coordinate plane having the same first coordinate or the same second coordinate; they will have done this graphically, because in grade 6 they did not have a general principle for finding such distances. In grade 7, they will use subtraction in such a general principle.

- a. Day 11. In contextual settings, students find the distance between two numbers on the number line graphically, in integer cases as well as in cases such as $-4 \frac{1}{3}$ and $-2/3$ that may induce them to connect the problem of finding distance to the operation of subtraction.

Students realize, or it is pointed out to them, that **the distance is found by taking the absolute value of the difference between the two numbers in either order, $|a - b|$, or, by subtracting the lesser number from the greater.** This can then be justified by appeal to models or definitions.

Students show, or are shown, that this principle agrees with distances found on the number line in earlier grades. Students also realize, or it is pointed out to them, that this principle is consistent with what they learned in Grade 6: at that time they learned, as a matter of definition, that the distance between 0 and a is $|a|$. The new principle is consistent, because it says that the distance between 0 and a is given by subtraction as $|a - 0| = |a|$.

- b. Day 12: No new mathematics is introduced. Students apply the subtraction principle to solve word problems involving the distance between two numbers, such as, a weather balloon is 100,000 feet above sea level, and a submarine is 3 miles below sea level. How high above the submarine is the weather balloon? John was \$3.75 in debt, and Mary was \$0.50 ahead. John found an envelope with some money in it, and after that he had the same amount of money as Mary. How much was in the envelope? On their way to giving the answer, students should represent these problems by a math diagram, a number line, and an equation.

- c. Day 13: Students apply the distance principle in word problems to find distances in the coordinate plane computationally for points with the same first coordinate or second coordinate, as they did graphically in grade 6.
 - d. Day 14: **Mixed practice.** Students practice problems along the lines of 2(e), 3(d), 4(b), 4(c).
5. Day 15: Unit test combining 2(e), 3(d), 4(b), 4(c). A score of 90% is required to pass.

Further developments during Grade 7

From this point on, algebra work with expressions and equations should sometimes involve positive and negative rational numbers, including positive and negative fractions and decimals, as well as parentheses. Also, applied percentage problems should sometimes involve negative percent differences.